BADLY APPROXIMABLE AFFINE FORMS AND SCHMIDT GAMES

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ABSTRACT. For any real number θ, the set of all real numbers x for which there exists a constant $c(x) > 0$ such that $\inf_{p \in \mathbb{Z}} |\theta q - x - p| \geq \frac{c(x)}{|q|}$ for all $q \in \mathbb{Z}\backslash\{0\}$ is an 1/8-winning set.

1. Introduction

Let $M_{m,n}(\mathbb{R})$ denote the set of $m \times n$ real matrices and $\widetilde{M}_{m,n}(\mathbb{R})$ denote $M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$. The element in $\widetilde{M}_{m,n}(\mathbb{R})$ corresponding to $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$ will be expressed as $\langle A, \mathbf{b} \rangle$. Consider the following well-known sets from the theory of Diophantine approximation [7]:

$$\mathbf{Bad}(m,n) := \left\{ \langle A, \mathbf{b} \rangle \in \widetilde{M}_{m,n}(\mathbb{R}) \mid \exists\ c(A,\mathbf{b}) > 0 \text{ s.t. } \|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} \geq \frac{c(A,\mathbf{b})}{\|\mathbf{q}\|^{n/m}}\ \forall \mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\} \right\}$$

where $\|\cdot\|$ is the sup norm on $\mathbb{R}^k$ and $\|\cdot\|_{\mathbb{Z}}$ is the norm on $\mathbb{R}^k$ given by $\|\mathbf{x}\|_{\mathbb{Z}} := \inf_{\mathbf{p} \in \mathbb{Z}^k} \|\mathbf{x} - \mathbf{p}\|$. The set $\mathbf{Bad}(m,n)$ is called the **set of badly approximable systems of m affine forms in n variables.** For any $\mathbf{b} \in \mathbb{R}^m$, let $\mathbf{Bad}^{\mathbf{b}}(m,n) := \{A \in M_{m,n}(\mathbb{R}) \mid \langle A, \mathbf{b} \rangle \in \mathbf{Bad}(m,n)\}$, and, for any $A \in M_{m,n}(\mathbb{R})$, let $\mathbf{Bad}_A(m,n) := \{\mathbf{b} \in \mathbb{R}^m \mid \langle A, \mathbf{b} \rangle \in \mathbf{Bad}(m,n)\}$.

The set $\mathbf{Bad}^{\mathbf{0}}(m,n)$ is called the **set of badly approximable systems of m linear forms in n variables** and is an important and classical object of study in the theory of Diophantine approximation. Although it is a Lebesgue null set (Khintchine, 1926), it has full Hausdorff dimension and, even stronger, is winning (Schmidt, 1969). Winning sets have a few other properties besides having full Hausdorff dimension; see Subsection 1.2 for more details.

For the larger set $\mathbf{Bad}(m,n)$, however, less is known. Among its known properties are that it has Lebesgue measure zero, but full Hausdorff dimension. The former property follows from the doubly metric inhomogeneous Khintchine-Groshev Theorem ([3], Chapter VII, Theorem II). The latter property is a result of D. Kleinbock (1999) proved using mixing of flows on the space of lattices [7]. Recently (2008), Y. Bugeaud, S. Harrap, S. Kristensen, and S. Velani have given a simpler proof of Kleinbock's result; their main result is that, for every A, $\mathbf{Bad}_A(m,n)$ (and some related sets) has full Hausdorff dimension [2]. Using the Marstrand slicing theorem ([4], Theorem 5.8), Kleinbock's result follows. In view of these results, a natural question that arises is whether, like $\mathbf{Bad}^{\mathbf{0}}(m,n)$, these sets $\mathbf{Bad}_A(m,n)$ and $\mathbf{Bad}(m,n)$ are winning instead of just having full Hausdorff dimension. In this note, we show that $\mathbf{Bad}_\theta(1,1)$ is winning for every real number θ.[1] For results and open questions concerning general n and m, see Remark 2.3 below.

[1]For $\mathbf{Bad}_\theta(1,1)$, we have a slight strengthening of the aforementioned consequence of the Khintchine-Groshev Theorem: $\mathbf{Bad}_\theta(1,1)$ has Lebesgue measure zero for every irrational number θ [6]. This result is essentially a corollary of two elementary facts from the theory of continued fractions (see [9] for this short, second proof and for a connection with shrinking targets). There is yet a third proof of this result; see [1].

1.1. **Statement of results.** Our main result, which generalizes the $m = n = 1$ case of the aforementioned main result in [2] (their main result is Theorem 1 of [2]), is the following (see Subsection 1.2 for the definition of 1/8-winning):

THEOREM 1.1. *For any real number θ, $\mathbf{Bad}_\theta(1,1)$ is an $1/8$-winning set.*

This theorem is proved in Section 2 below. A number of corollaries will follow immediately because of properties of winning sets (see Subsection 1.2). A model one is

COROLLARY 1.2. *For any countable set $\{\theta_n\} \subset \mathbb{R}$ and any countable family $\{f_m\}$ of invertible affine maps $\mathbb{R} \to \mathbb{R}$, the set $\cap_{m=1}^\infty \cap_{n=1}^\infty f_m(\mathbf{Bad}_{\theta_n}(1,1))$ is $1/8$-winning and thus has full Hausdorff dimension.*

1.2. **Background on winning sets and continued fractions.** The proof of our result requires two tools: Schmidt games (see [8] for a reference) and continued fractions (see [5] for a reference). We will discuss both.

W. Schmidt introduced the games which now bear his name in [8]. Let $0 < \alpha < 1$ and $0 < \beta < 1$. Let S be a subset of a complete metric space M. Two players, Black and White, alternate choosing nested closed balls $B_1 \supset W_1 \supset B_2 \supset W_2 \cdots$ on M. The radius of W_n must be α times the radius of B_n, and the radius of B_n must be β times the radius of W_{n-1}. The second player, White, wins if the intersection of these balls lies in S. A set S is called (α, β)**-winning** if White can always win for the given α and β. A set S is called α**-winning** if White can always win for the given α and any β. A set S is called **winning** if it is α-winning for some α. Schmidt games have four important properties for us [8]:

- The sets in $\mathbb{R}^n$ which are α-winning have full Hausdorff dimension.
- Countable intersections of α-winning sets are again α-winning.
- The bilipschitz image of an α-winning set is α-winning.
- Let $0 < \alpha \leq 1/2$. If a set in a Banach space of positive dimension is α-winning, then the set with a countable number of points removed is also α-winning.

Let us now discuss continued fractions. Let p_i/q_i be the i-th order convergent of an irrational number θ. Define

$$\Delta_i := \|\theta q_i\|_{\mathbb{Z}}.$$

We will use the following well-known facts:

- For all $i \in \mathbb{N}$, $\frac{1}{2}\Delta_{i-1}^{-1} < q_i \leq \Delta_{i-1}^{-1}$.
- Let $0 \leq j < k < q_i$. Then, $\|\theta k - \theta j\|_{\mathbb{Z}} \geq \Delta_{i-1}$.

1.3. **The setup.** Let $\theta \in \mathbb{R}$. Define

$$\mathbf{Bad}_\theta^+ := \Big\{x \in \mathbb{R} \mid \exists\ c(x) > 0 \text{ s.t. } \|\theta q - x\|_{\mathbb{Z}} \geq \frac{c(x)}{q}\ \forall q \in \mathbb{N}\Big\}.$$

Note that $\mathbf{Bad}_\theta(1,1) = \mathbf{Bad}_\theta^+ \cap -\mathbf{Bad}_\theta^+$; thus showing $\mathbf{Bad}_\theta^+$ is 1/8-winning will prove Theorem 1.1. Also, we may assume that these sets are restricted to the circle $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$, as they are invariant under integral translations.

Henceforth, let us consider $\mathbf{Bad}_\theta^+$. If θ is rational, then the set is just $\mathbb{T}^1$ with a finite number of points removed and hence is winning. Therefore, we assume that θ is irrational henceforth.

For convenience, let us call the elements in

$$\{\theta q \in \mathbb{T}^1 \mid q_i \leq q < q_{i+1}\}$$

the **elements of generation** i**.**
Finally, we note a simple property of continued fractions:

LEMMA 1.3. *Let* $q_{i+1} \leq q < q_{i+2}$. *Given a* $0 < r < 1/2$ *such that, for all elements* θp *of generations* $\leq i$, $\|\theta q - \theta p\|_{\mathbb{Z}} \geq r\Delta_i$, *then* $q \geq \frac{r}{2} q_{i+2}$.

Proof. There are unique numbers $0 \leq s < q_{i+1}$ and $1 \leq n \leq \lfloor \frac{q_{i+2}}{q_{i+1}} \rfloor$ such that $q = nq_{i+1} + s$. Thus, $n\Delta_{i+1} = \|\theta q - \theta s\|_{\mathbb{Z}} \geq r\Delta_i$. Hence, $q \geq r\frac{\Delta_i}{\Delta_{i+1}} q_{i+1} \geq \frac{r}{2} q_{i+2}$. □

2. A PROOF OF THEOREM 1.1

Let $\alpha = 1/8$ and $c = (\frac{(\alpha\beta)}{4})^3$. We will play an (α, β)-game on $\mathbb{T}^1$. Let us start with the following lemma, which tells us how to choose W_m given B_m (note that the radius of a ball B will be denoted $\rho(B)$):

LEMMA 2.1. *Let* U *be any union of balls on* $\mathbb{T}^1$ *with radius* $\leq (\alpha\beta)\Delta_N/4$ *around the elements of generations* $\leq N$. *If*

$$(\alpha\beta)\Delta_N < 2\rho(B_m) \leq \Delta_N,$$

then one can choose W_m *disjoint from* U.

Proof. **Case:** B_m **does not intersect any ball of** U**.**
Pick any allowed W_m.

Case: B_m **intersects exactly one ball of** U**.**
Even if B_m contains the whole ball of U, there is, at least, a subinterval in B_m of length 1/4 of the length of B_m that misses U. Pick W_m to be in this subinterval.

Case: B_m **intersects more than one ball of** U**.** Note that B_m cannot intersect more than one element of generations $\leq N$ (unless one has exactly two elements of generations $\leq N$, one at each end). Thus, at least a subinterval in B_m of length $(1-(\alpha\beta)/2)\Delta_N \geq 1/2\Delta_N$ does not meet U. Now $\alpha 2\rho(B_m) \leq 1/8\Delta_N$. Therefore, we can choose W_m to be in this subinterval. □

Since the Schmidt game can be played until, for some $J \in \mathbb{N}$, $2\rho(B_J) \leq \Delta_1$, we may assume without loss of generality that $J = 1$. Note that there exists a $N \geq 2$ such that $2\rho(B_1) \leq \Delta_{N-1}$, but that $2\rho(B_1) > \Delta_N$ (follows since $\Delta_N < \Delta_{N-1}$).

Also, there exists a $n_0 \in \mathbb{N}$ such that $2(\alpha\beta)^{n_0-1}\rho(B_1) > \Delta_N$ and $2(\alpha\beta)^{n_0}\rho(B_1) \leq \Delta_N$. Thus,

$$(\alpha\beta)\Delta_N < 2(\alpha\beta)^{n_0}\rho(B_1) \leq \Delta_N. \tag{2.1}$$

We require that N be the largest natural number for which (2.1) holds.

We intend to use induction. In the initial induction step, consider the disjoint union of balls around each element of generations $\leq N$ of radius $(\alpha\beta)\Delta_N/4$; call this union U. By Lemma 2.1, we may pick W_{n_0+1} to miss U. For any other step of the induction, W_{n_0+1} is already chosen.

2.1. **Case:** $\alpha\beta\Delta_N > \Delta_{N+1}$**.** The condition implies that there exists a $n_1 \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+1} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leq \Delta_{N+1}.$$

Also, there exists a maximal $M \geq 1$ such that

$$(\alpha\beta)\Delta_{N+M} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leq \Delta_{N+M}.$$

Moreover, $(\alpha\beta)\Delta_{N+1} < \Delta_{N+M}$.

For any element θq of generation $N+1$ in W_{n_0+1}, $q \geq \frac{(\alpha\beta)}{8}q_{N+2}$ by Lemma 1.3. For any element θq of generations $> N+1$ in W_{n_0+1}, it is obvious that $q \geq \frac{(\alpha\beta)}{8}q_{N+2}$. Thus, for all such θq,

$$\frac{c}{q} \leq \frac{(\alpha\beta)^2\Delta_{N+1}}{4} \leq \frac{(\alpha\beta)\Delta_{N+M}}{4}.$$

Now play freely until $B_{n_0+n_1+1}$ is chosen. Again by Lemma 2.1, we can choose $W_{n_0+n_1+1}$ to miss the balls of radius $(\alpha\beta)\Delta_{N+M}/4$ around the elements of generations $N+1$ to $N+M$.

2.2. **Case:** $\alpha\beta\Delta_N \leq \Delta_{N+1}$**.** It is easy to see from the theory of continued fractions that there exist a $K \in \mathbb{N}$ such that $(\alpha\beta)\Delta_n > \Delta_{n+K}$ for all $n \in \mathbb{N}$. Therefore, the condition implies that there exists a $1 \leq m \leq K-1$ such that

$$\Delta_{N+m+1} < \alpha\beta\Delta_N \leq \Delta_{N+m}.$$

Thus, we have

$$(\alpha\beta)^2\Delta_{N+m} < (\alpha\beta)^2\Delta_N < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leq \alpha\beta\Delta_N \leq \Delta_{N+m}.$$

If $(\alpha\beta)^2\Delta_{N+m} < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leq (\alpha\beta)\Delta_{N+m}$, then

$$(\alpha\beta)\Delta_{N+m} < 2(\alpha\beta)^{n_0}\rho(B_1) \leq \Delta_{N+m}.$$

Since N is the largest natural number for which (2.1) holds, we obtain that $m = 0$, a contradiction.

Thus, we must conclude that

$$(\alpha\beta)\Delta_{N+m} < 2(\alpha\beta)^{n_0+1}\rho(B_1) \leq \Delta_{N+m}.$$

Now, there exists a $n_1 \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+m+1} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leq \Delta_{N+m+1}.$$

Also, there exists a maximal $M \in \mathbb{N}$ such that

$$(\alpha\beta)\Delta_{N+m+M} < 2(\alpha\beta)^{n_0+n_1}\rho(B_1) \leq \Delta_{N+m+M}.$$

Moreover, $(\alpha\beta)\Delta_{N+m+1} < \Delta_{N+m+M}$.

If $n_1 = 1$, then even more is true: $(\alpha\beta)\Delta_{N+m} < \Delta_{N+m+M}$. Now note that, for the elements θq of generations $N+1$ to $N+m+M$, we have

$$\frac{c}{q} \leq \frac{c}{q_{N+1}} \leq \frac{(\alpha\beta)\Delta_{N+m+M}}{4}.$$

Consider the disjoint union of balls around each element of generations $\leq N+m+M$ of radius $(\alpha\beta)\Delta_{N+m+M}/4$; call this union U. Again by Lemma 2.1, we can pick W_{n_0+2} to miss U.

Otherwise, $n_1 \geq 2$. Now note that, for the elements θq of generations $N+1$ to $N+m$, we have

$$\frac{c}{q} \leq \frac{c}{q_{N+1}} \leq \frac{(\alpha\beta)\Delta_{N+m}}{4}.$$

Consider the disjoint union of balls around each element of generations $\leq N+m$ of radius $(\alpha\beta)\Delta_{N+m}/4$; call this union U. Again by Lemma 2.1, we can pick W_{n_0+2} to miss U.

For any element θq of generation $N+m+1$ in W_{n_0+2}, $q \geq \frac{(\alpha\beta)}{8} q_{N+m+2}$ by Lemma 1.3. For any element θq of generations $> N+m+1$ in W_{n_0+2}, it is obvious that $q \geq \frac{(\alpha\beta)}{8} q_{N+m+2}$. Thus, for all such θq,

$$\frac{c}{q} \leq \frac{(\alpha\beta)^2 \Delta_{N+m+1}}{4} \leq \frac{(\alpha\beta)\Delta_{N+m+M}}{4}.$$

Now play freely until $B_{n_0+n_1+1}$ is chosen. Again by Lemma 2.1, we can choose $W_{n_0+n_1+1}$ to miss the balls of radius $(\alpha\beta)\Delta_{N+m+M}/4$ around the elements of generations $N+m+1$ to $N+m+M$.

Using these two cases inductively, one can show that the set

$$\left\{x \in \mathbb{R} \mid \exists\; c(x) > 0 \text{ s.t. } \|\theta q - x\|_{\mathbb{Z}} \geq \frac{c(x)}{q}\ \forall q \geq q_{N+1}\right\}$$

is 1/8-winning. By shrinking $c(x)$ for each x, we note that this set is $\mathbf{Bad}_\theta^+$. The proof is complete.

Remark 2.2. If θ is a badly approximable number[2], one can easily see from the continued fraction expansion of θ that there exists an upper bound for Δ_n/Δ_{n+1} independent of n. This uniform bound allows us to simplify the above proof for θ badly approximable (however, we conclude that the set is α-winning for an α depending on this uniform bound).

Remark 2.3. It should be noted that Manfred Einsiedler and the author, in joint work in preparation, are able to generalize Theorem 1 of [2], using a method different from the one presented in this note, to conclude winning, instead of just having full Hausdorff dimension. Thus, as a special case, we can show that $\mathbf{Bad}_A(m,n)$ is winning for every $A \in M_{m,n}(\mathbb{R})$. Whether $\mathbf{Bad}(m,n)$ is winning, however, is still an open question. See Section 5 of [7] for a list of this and other open questions related to winning.

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[2]In our notation, $\theta \in \mathbf{Bad}^0(1,1)$.

References

[1] V. Beresnevich, V. Bernik, M. Dodson, and S. Velani, *Classical metric Diophantine approximation revisited,* preprint, arXiv:0803.2351v1 (2008).

[2] Y. Bugeaud, S. Harrap, S. Kristensen, and S. Velani, *On shrinking targets for $\mathbb{Z}^m$ actions on tori,* preprint, arXiv:0807.3863v1 (2008).

[3] J. Cassels, "An Introduction to Diophantine Approximation," Cambridge Tracts in Mathematics and Mathematical Physics **45**, Cambridge University Press, Cambridge, UK, 1957.

[4] K. Falconer, "The geometry of fractal sets," Cambridge Tracts in Mathematics, **85**, Cambridge University Press, Cambridge, 1986.

[5] A. Khinchin, "Continued Fractions," The University of Chicago Press, Chicago, 1964.

[6] D. Kim, *The shrinking target property of irrational rotations*, Nonlinearity **20** (2007), 1637–1643.

[7] D. Kleinbock, *Badly approximable systems of affine forms*, J. Number Theory **79** (1999), 83 –102.

[8] W. Schmidt, *Badly approximable numbers and certain games,* Trans. Amer. Math. Soc. **123** (1966), 178–199.

[9] J. Tseng, *On circle rotations and the shrinking target properties,* Discrete Contin. Dyn. Syst. **20** (2008), 1111-1122.

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